Exercise 7.2.9

Prove that Eq. (7.12) is exact in the sense of Eq. (7.9), provided that $\alpha(x)$ satisfies Eq. (7.14).

Solution

Eq. (7.12) is on page 336.

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x) \tag{7.12}$$

Here $\alpha(x)$ is an integrating factor that's been multiplied by both sides of a linear first-order inhomogeneous ODE. It satisfies

$$\frac{d\alpha}{dx} = \alpha(x)p(x). \tag{7.14}$$

Bring all terms in Eq. (7.12) to the left side.

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y - \alpha(x)q(x) = 0$$

Multiply both sides by dx.

$$[\alpha(x)p(x)y - \alpha(x)q(x)]\,dx + [\alpha(x)]\,dy = 0$$

For this ODE to be exact, it must satisfy the condition in Eq. (7.9).

$$\begin{aligned}
\frac{\partial}{\partial y}[\alpha(x)p(x)y - \alpha(x)q(x)] &\stackrel{?}{=} \frac{\partial}{\partial x}[\alpha(x)] \\
\alpha(x)p(x) &\stackrel{?}{=} \frac{d\alpha}{dx} \\
\alpha(x)p(x) &= \alpha(x)p(x)
\end{aligned}$$

Because this condition is satisfied, the ODE in Eq. (7.12) is exact, and there exists a potential function $\varphi = \varphi(x, y)$ such that

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &= \alpha(x)p(x)y - \alpha(x)q(x) \\
\frac{\partial \varphi}{\partial y} &= \alpha(x).
\end{aligned}$$